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Robust Flight Control System Design with Multiple Model Approach

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Introduction

CONTROL system design for uncertain dynamical systems and systems with changing parameters has been one of the major research subjects in linear control theory. Recently, new theoretical results have been introduced using the frequency response, such as the H_∞ approach, and these results have been attempted to be extensively applied to flight control problems. The multiple model approach discussed here has been developed with the same objective (e.g., Ref. 1). The plant dynamics is not uniquely given, but it is described using multiple candidates of dynamical systems or multiple models.

The concept of multiple models has long been used in flight control system design. Changes of parameters, such as dynamic pressure, Mach number, weight and balance, and configuration, have a significant influence on the dynamic properties of aircraft, and consideration of every necessary point in the flight envelope is important for the design of a flight control system. Although such design attempts have often been carried out in a trial-and-error manner based on empirical knowledge, an extension of control theory with the multiple model approach has been proposed for more efficient design. The multiple model approach has been combined with linear quadratic regulator (LQR),[2,3] linear quadratic Gaussian (LQG),[3-5] and pole assignment.[1]

For flight control system design, not only prescribed changes of parameters but also uncertain dynamics should be considered. Uncertainty in the high frequency range, which derives from various sources such as flexible modes, nonlinearity of the actuator system, uncertain time lags due to digital signal processing, and the effects of filters, is frequently pointed out for careful consideration. In practice, designers often find that high gain feedback control introduces a high crossover frequency, where phase information is quite uncertain, and this makes the closed-loop system unstable.[6] This is one of the major motivations of recent robust control research. In the robust control problems, how to present uncertainty of the dynamical system is important. In the present Note, an uncertain delay model is proposed to present uncertainty in the high frequency range. The multiple model approach with an uncertain delay model can introduce the maximum regulator performance while suppressing the frequency bandwidths of feedback control. By considering different times of delay, bandwidths can be adjusted for different points of the closed-loop system.

The present multiple model approach was successfully applied to an active flutter control problem.[7] The approach is applied to a flight control problem emphasizing regulator performance and bandwidth for the multi-input control system. A prefixed control structure, such as proportional output feedback with a fixed gain, can provide a practical control law, and it is useful to avoid acute dependency on the selected points for multiple models.[8] Numerical results show practical feasibility of the approach for flight control system design.

Multiple Model Approach

The LQR methodology is extended to the multiple model problem while posing a constraint of proportional output feedback. The following is a brief review of the approach. The plant dynamics is given by the following multiple models:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = A_i x_i(t) + B_i u_i(t), \qquad y_i(t) = C_i x_i(t); \qquad i = 1, M \tag{1}$$

where $x_i \in R^{n_i}$ $u_i \in R^m$, $y_i \in R^r$ are the state variable, control variable, and output variable, respectively. The subscript i denotes the ith model, and the system matrices (A_i, B_i, C_i) are constant matrices of adequate size. The initial condition of state variables is considered random, and its characteristic is defined as

$$E[x_i(0)] = 0, \qquad E[x_i(0)x_i^T(0)] = W_i; \qquad i = 1, M \tag{2}$$

where $E[\ ]$ denotes the average, and $W_i(\geq 0)$ is appropriately given. The control law is defined as follows:

$$u_i(t) = K y_i(t) \tag{3}$$

The feedback gain K is common to all models. The output y_i may be a variable that is directly measured or a variable that is accurately reconstructed with a filter.

When the closed-loop systems, $A_i + B_i K C_i$, are asymptotically stable, the performance index is defined as a weighted summation of performance indices for each model, i.e.,

$$J = \sum_{i=1}^{M} p_i J_i, \quad J_i = E\left[\int_0^\infty x_i(t)^T Q_i x_i(t) + u_i(t)^T R u_i(t)\, \mathrm{d}t\right] \tag{4}$$

where $p_i(>0, \sum p_i = 1)$ is the probability of the ith model, or it can be used as an adjustable design parameter. Weighting matrices $Q_i(\geq)$ and $R(>0)$ are appropriately given as the standard LQR problem. The performance index can be rewritten as

$$J = \sum_{i=1}^{M} p_i \operatorname{tr}[(Q_i + C_i^T K^T R K C_i) X_i] \tag{5}$$

$$(A_i + B_i K C_i) X_i + X_i (A_i + B_i K C_i)^T + W_i = 0$$

$$X_i = E\left[\int_0^\infty x_i(t) x_i(t)^T\, \mathrm{d}t\right] \tag{6}$$

The optimal gain that minimizes the performance index satisfies the following necessary conditions or the optimality conditions:

$$\sum_{i=1}^{M} p_i R K_i C_i X_i C_i^T + p_i B_i^T Y_i X_i C_i^T = 0 \tag{7}$$

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$$(A_i + B_iKC_i)^T Y_i + Y_i(A_i + B_iKC_i) + Q_i + C_i^T K^T RKC_i = 0 \quad (8)$$

where $Y_i \in R^{n_i \times n_i}$, $i = 1, M$, are Lagrange multipliers for the constraint (6).

Although there is no direct computational method for the optimal solution, the gradient of the performance index with respect to the feedback gain K is easily calculated; therefore, any kind of gradient method can be applied to this problem. An exception is the case where there is no initial feedback gain that stabilizes all models. A penalty function method that was proposed by Miyazawa and Dowell[7] is suitable for such a case. The algorithm uses the following performance index:

$$J^* = \sum_{i=1}^{M} p_i \operatorname{tr}[(Q_i + K_i^T RK_i)X_i + \mu(K_i - KC_i)^T R(K_i - KC_i)X_i] \quad (9)$$

where $(A_i + B_iK_i)X_i + X_i(A_i + B_iK_i)^T + W_i = 0$. The constraint of output feedback is relaxed to full state feedback with different gains for each model, i.e., $u_i(t) = K_ix_i(t)$, and the penalty is posed for the constraint with the last term in the performance index (9). The penalty parameter is $\mu(>0)$. A major characteristic of this algorithm is that the closed-loop system $(A_i + B_iK_i)$ is asymptotically stable in any case, and it makes the iterative calculation more efficient. Details of the algorithm were discussed in Ref. 7. The numerical example in the present Note was calculated with the penalty function method, and the quasi-Newton method was also used to improve the final accuracy.

Robust Control System with Time Delay Element

The first-order delay element given by the following transfer function is proposed to present the uncertainty of the plant dynamics in the high frequency range:

$$\frac{u^*}{u} = \frac{1 - Ts/2}{1 + Ts/2} \quad (10)$$

This is the [1, 1] Padé approximation of the pure time delay of time T s. The frequency response has a unity gain for any frequency, but its phase lag increases along with increased frequency. At a frequency of $\omega = 2/T$ rad/s, it has a 90-deg phase lag, and in the high frequency range it approaches an asymptote of 180-deg phase lag. Two models that are $T_1 = 0$ and $T_2 = 2/\omega_0(>0)$ are adequate to present system uncertainty for the high frequency range, where ω_0 could be called the boundary frequency. The phase difference is almost 180 deg in the high frequency range, and it means that the signs of the responses are opposite. Roughly speaking, ω_0 can be about the upper limit of the crossover frequency at the point where the uncertain delay model is inserted. Although the first-order time delay element in Eq. (10) does not claim any optimality for presenting uncertain dynamics in the high frequency range, it is one of the most simple dynamics, and it can be applied to most general cases.

Application of Multimodel Approach to Longitudinal Flight Control

As a numerical example, a robust flight control system is designed for a landing approach condition of a twin-turboprop light transport. For a landing approach, precise path control and airspeed regulation are the most important. To accomplish this design objective, two control variables, elevator and power, can be used for the longitudinal motion. Although the frequency bandwidths of these control variables differ with each other, the multimodel approach can introduce an appropriate control law in such a case.

The longitudinal motion of the airplane is described by the following linear time-invariant state equation[9]:

$$\frac{dx}{dt} = Ax(t) + Bu(t), \qquad y(t) = Cx(t) \quad (11)$$

$$x^T = [u(\mathrm{m/s}),\ \alpha(\mathrm{rad}),\ q(\mathrm{rad/s}),\ \theta(\mathrm{rad}),\ h(\mathrm{m})]$$

$$u^T = [\delta_e(\mathrm{rad}),\ \delta_T]$$

$$y^T = [u(\mathrm{m/s}),\ q(\mathrm{rad/s}),\ \theta(\mathrm{rad}),\ \frac{dh}{dt}(\mathrm{m/s}),\ h(\mathrm{m})]$$

$$A = \begin{bmatrix} -0.0585 & 4.75 & 0. & -9.80 & 0. \\ -0.00978 & -0.960 & 0.985 & 0.0115 & 0. \\ 0.00372 & -1.79 & -1.49 & -0.0044 & 0. \\ 0. & 0. & 1. & 0. & 0. \\ 0. & -44.75 & 0. & 44.75 & 0. \end{bmatrix}$$

$$B = \begin{bmatrix} 0. & 7.01 \\ -0.0533 & -0.107 \\ -2.11 & 0.232 \\ 0. & 0. \\ 0. & 0. \end{bmatrix}, \quad C = \begin{bmatrix} 1. & 0. & 0. & 0. & 0. \\ 0. & 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 1. & 0. \\ 0. & -44.75 & 0. & 44.75 & 0. \\ 0. & 0. & 0. & 0. & 1. \end{bmatrix}$$

where u is the true airspeed, α the angle of attack, q the pitch rate, θ the pitch angle, h the vertical deviation from the reference flight path, dh/dt the vertical speed, δ_e the elevator angle, and δ_T the thrust coefficient. All variables except q are deviations from nominal values of the equilibrium condition. The flight condition for this example is gear down, 20 deg flap down, 3 deg glide slope angle, and $U_0 = 44.75$ m/s (87 kt), where U_0 is the true air speed of equilibrium condition. Concerning the measurement, it is assumed that all states are measured exactly, and the angle of attack α is replaced by vertical speed dh/dt to simplify the interpretation of the feedback gain.

Before the discussion of the multimodel approach, the problem is formulated using LQR. Since regulation of the deviations from the nominal flight path and true airspeed is the major concern, the performance index can be defined as

$$J = U + H + r_e\Delta_e + r_T\Delta_T$$

$$U = E\left[\int_0^\infty \left(\frac{u}{u_0}\right)^2 dt\right], \qquad H = E\left[\int_0^\infty \left(\frac{h}{h_0}\right)^2 dt\right]$$

$$\Delta_e = E\left[\int_0^\infty \left(\frac{\delta_e}{\delta_{e_0}}\right)^2 dt\right], \qquad \Delta_T = E\left[\int_0^\infty \left(\frac{\delta_T}{\delta_{T_0}}\right)^2 dt\right] \quad (12)$$

where u_0, h_0, δ_{e_0}, and δ_{T_0} are appropriately defined nominal values.[10] Here they are simply given as $u_0 = 4.48$ m/s ($= U_0/10$), $h_0 = 5$ m, $\delta_{e_0} = 0.2$ rad, and $\delta_{T_0} = 0.1$. Weighting parameters of the control inputs r_e and r_T may be considered as adjustable parameters to realize the bandwidths of each control input. In general, definition of the weighting matrices Q and R in the performance index is free for designers, and it may be used to enhance robustness of the control system. In the present approach, however, the quadratic performance index is defined as simply as possible to directly present the design objective, and robustness against uncertain dynamics is guaranteed with the multiple model approach.

The control law of the multimodel approach is introduced, considering uncertain delay elements inserted into each control input, as shown in Fig. 1. The dynamics for the elevator actuation and thrust generation are not explicitly considered, but they are included in the uncertainty. Four models, which consist of two cases of delay time for the two inputs, are taken into account, i.e.,

$$T_{e_1} = T_{e_3} = 0, \qquad T_{e_2} = T_{e_4} = 0.5 \text{ s}$$

$$T_{T_1} = T_{T_2} = 0, \qquad T_{T_3} = T_{T_4} = 2 \text{ s} \quad (13)$$

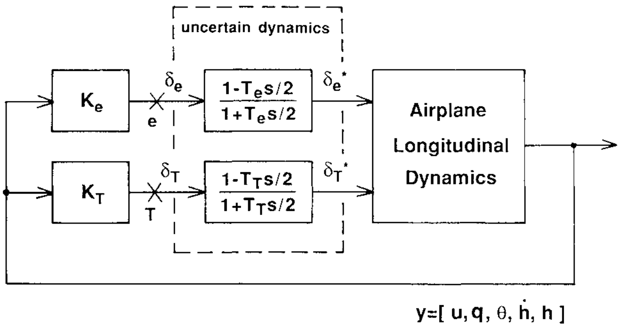

Fig. 1 Block diagram of longitudinal flight control system at a landing approach condition.

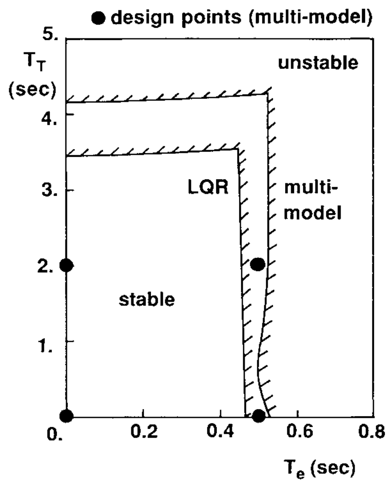

Fig. 2 Stability vs delay times.

where T_{e_i} and T_{T_i} are the delay times for the inputs of elevator and thrust, the subscript i denotes the model number, and $T_e = 0.5$ s and $T_T = 2$ s correspond to 4 rad/s and 1 rad/s of boundary frequencies, respectively. These figures are roughly chosen for the numerical example. Generally, thrust generation with the power control is slower and more uncertain than elevator actuation. The performance index for each model is the same as previously defined in Eq. (12), and the total performance index for the multiple model system is simply defined as

$$J = \sum_{i=1}^{4} \frac{1}{4} J_i, \quad J_i = U_i + H_i + r_e \Delta_{e_i} + r_T \Delta_{T_i}; \quad i = 1, 4 \tag{14}$$

where U_i, H_i, Δ_{e_i}, and Δ_{T_i} are defined as Eq. (12), and subscript i denotes a variable of the ith model. To obtain maximum regulator performance, the control cost is omitted from the performance index, i.e., $r_e = r_T = 0$. The maximum frequency bandwidth is limited due to the uncertain delay element, and the feedback gain is finite even when the weighting on the control cost is zero. Although rigorous discussion of the existence of a finite solution is open to question, in the numerical calculation, the feedback gain is obtained for the case of $r_e = r_T = 0$. With respect to the initial condition, the optimal feedback gain of the multimodel depends on the matrix W_i. As well as the weighting matrices Q_i and R, W_i should be simply given in a natural manner. For this numerical example, $W_i = \mathrm{diag}[u_0^2, 0, 0, 0, h_0^2]$ is given.

The optimal feedback gain is obtained as follows:

$$K = \begin{bmatrix} -0.0129 & 2.29 & 3.05 & 0.186 & 0.144 \\ -0.0637 & -0.278 & -0.687 & -0.0567 & -0.0537 \end{bmatrix} \tag{15}$$

The performance indices for each model are obtained as $J_1 = 2.22$, $J_2 = 2.35$, $J_3 = 4.16$, and $J_4 = 4.49$. For the first model, which has no delay in the loops, the performance index and control costs are broken down as $U_1 = 0.967$, $H_1 = 1.253$, $\Delta_{e_1} = 1.633$, and $\Delta_{T_1} = 10.59$. The crossover frequencies and phase margins at break points e and T, which are indicated by x in Fig. 1, are $\omega_{ce} = 3.49$ rad/s, $PM_e = 88$ deg, and $\omega_{cT} = 0.50$ rad/s, $PM_T = 89$ deg. These figures are for the no-delay case.

To compare the control law of the multimodel approach with the standard LQR method, the same control cost is searched in the LQR (12) under the condition of no-delay time by adjusting two weighting parameters r_e and r_T. When $r_e = 0.1043$ and $r_T = 0.0830$, the LQR gives the following quadratic performances as $U = 0.856$, $H = 1.290$, $\Delta_e = 1.633$, and $\Delta_T = 10.59$. The regulator performance is quite similar to that of the multimodel approach. The crossover frequencies and phase margins of the LQR at break points e and T become $\omega_{ce} = 2.54$ rad/s, $PM_e = 60$ deg, and $\omega_{cT} = 0.57$ rad/s, $PM_T = 88$ deg. These figures show that the multimodel approach gives larger crossover frequency and phase margin for the elevator input channel than the standard LQR method. For the power control channel, the multimodel approach gives a little smaller crossover frequency but nearly the same phase margin with the LQR. Figure 2 shows the stability region for both control laws against the change of delay times in the two control inputs. This stability boundary map also demonstrates that the multiple model approach can give a more robust control law than the standard LQR method with a small sacrifice in regulator performance. Although the present feedback gain obtained with the multimodel approach (15) does not claim the final control law to be implemented in the real flight control system, the numerical result could demonstrate that the multimodel approach can be a strong tool for the robust control system design.

Concluding Remarks

An approach that is characterized by three concepts—multiple models, LQR, and constraint of proportional output feedback—is applied to robust flight control system design. A first-order delay model is proposed to present the uncertain dynamics in the high frequency range. With this approach, a control law is obtained systematically by defining a small number of design parameters, including permissible bandwidths for each control input. Two aspects are important for the approach to be useful in practice. One is introduction of an appropriate uncertain model, such as the proposed delay model. The other is an efficient computational algorithm that can always produce a converged solution. The penalty function method is appropriate for that purpose.

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Optimal Transfers Between Coplanar Elliptical Orbits

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Optimal Transfer Equations

THE problem to be studied is that of optimal transfer of a rocket vehicle between a pair of coplanar elliptical orbits, employing two impulsive thrusts. The criterion for optimality is minimization of the propellant expenditure or, equivalently, of the characteristic velocity for the maneuver. If v_1 and v_2 denote the vector velocity increments generated in the vehicle by the impulsive thrusts, then the characteristic velocity V for the maneuver is defined by the equation

$$V = v_1 + v_2 \tag{1}$$

where v_1 and v_2 are the magnitudes of the increments. If c is the magnitude of the rocket jet velocity and R is the mass ratio for the maneuver, then it is well known that $c\ell nR = V$.

Taking the center of gravitational attraction as pole, the polar coordinates of the rocket will be denoted by $(1/u, \Theta)$, and it will be assumed that, for both terminal orbits, Θ increases with the time t. The polar equation of an orbit will be taken in the form

$$u = a + b\cos(\Theta - \omega) \tag{2}$$

where the constants a, b, and ω $(a > b > 0)$ will be termed the elements of the orbit. Denoting the semilatus rectum and the eccentricity of the orbit by ℓ and e, respectively, we have

$$\ell = 1/a, \qquad e = b/a \tag{3}$$

where ω is called the longitude of the periapse and is measured from a convenient reference line $\Theta = 0$.

The direction of an impulsive thrust I applied to a rocket situated at a point P $(1/u, \Theta)$ (Fig. 1) is determined by the angle ϕ it makes with the forward transverse direction (i.e., perpendicular to the radius vector OP). In the diagram, A is the periapse; ϕ is measured counterclockwise between 0 and 360 deg.

It has been proved[1] that, if the impulse transfers the rocket from an orbit (a, b, ω) into an orbit (a', b', ω'), then the characteristic velocity v for the transfer is given by

$$v = \gamma^{1/2} u (a'^{-1/2} - a^{-1/2}) \sec\phi \tag{4}$$

where γu^2 is the gravitational acceleration toward the center of attraction O. Thus, if (a_1, b_1, ω_1) and (a_2, b_2, ω_2) are the elements of the terminal orbits, and (a, b, ω) are the elements of the transfer orbit, then the characteristic velocity V for the maneuver is given by

$$V = \gamma^{1/2} u_1 (a^{-1/2} - a_1^{-1/2}) \sec\phi_1 + \gamma^{1/2} u_2 (a_2^{-1/2} - a^{-1/2}) \sec\phi_2 \tag{5}$$

where $(1/u_1, \Theta_1)$ and $(1/u_2, \Theta_2)$ are the coordinates of the two junction points and (ϕ_1, ϕ_2) specify the directions of the impulsive thrusts.

If w denotes the component of rocket velocity in a direction perpendicular to the thrust (see Fig. 1), which is unaffected by the thrust, we define A by the equation

$$A = \gamma^{-1/2} w \operatorname{cosec}\phi \tag{6}$$

Then, if (A_1, A_2) are the respective values of A for the impulsive thrusts I_1, I_2, it has been shown[1,2] that for V to be stationary with respect to variations in the elements of the transfer orbit, it is necessary that

$$\left(\frac{u_1 + a}{A_1 a^{1/2}} + 1\right) \cos\phi_1 = \left(\frac{u_2 + a}{A_2 a^{1/2}} + 1\right) \cos\phi_2 \tag{7}$$

$$\left(\frac{u_1}{A_1} - A_1\right) \sin\phi_1 = \left(\frac{u_2}{A_2} - A_2\right) \sin\phi_2 \tag{8}$$

$$(u_1 - a)\left(1 + \frac{a^{1/2}}{A_1}\right) \cos\phi_1 + (u_1 - A_1 a^{1/2}) \sin\phi_1 \tan\phi_1$$
$$= (u_2 - a)\left(1 + \frac{a^{1/2}}{A_2}\right) \cos\phi_2 + (u_2 - A_2 a^{1/2}) \sin\phi_2 \tan\phi_2 \tag{9}$$

In the same places, it was shown that A_1 and A_2 satisfy the equations

$$b_1 \sin(\Theta_1 - \omega_1) = (u_1 - A_1 a_1^{1/2}) \tan\phi_1 \tag{10}$$

$$b \sin(\Theta_1 - \omega) = (u_1 - A_1 a^{1/2}) \tan\phi_1 \tag{11}$$

$$b_2 \sin(\Theta_2 - \omega_2) = (u_2 - A_2 a_2^{1/2}) \tan\phi_2 \tag{12}$$

$$b \sin(\Theta_2 - \omega) = (u_2 - A_2 a^{1/2}) \tan\phi_2 \tag{13}$$

A further four equations are derived by substitution in the equations of the three orbits,

$$b_1 \cos(\Theta_1 - \omega_1) = u_1 - a_1 \tag{14}$$

$$b \cos(\Theta_1 - \omega) = u_1 - a \tag{15}$$

$$b_2 \cos(\Theta_2 - \omega_2) = u_2 - a_2 \tag{16}$$

$$b \cos(\Theta_2 - \omega) = u_2 - a \tag{17}$$

The last eleven equations are sufficient to determine sets of values for the 11 unknowns a, b, ω, u_1, u_2, Θ_1, Θ_2, A_1, A_2, ϕ_1, and ϕ_2 corresponding to stationary values of the characteristic velocity V [given by Eq. (5)]. Note that positive values of $a^{1/2}$, etc., are intended everywhere.

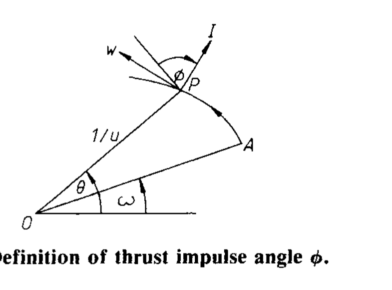

Fig. 1 Definition of thrust impulse angle ϕ.

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